



9.4 Matrices of orthonormal vectors and their inverses

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Introduction

- In this topic, we will
 - Define and consider permutation matrices
 - Try to determine the inverse of such permutation matrices
 - Introduce the concept of the transpose
 - Consider real matrices where the columns form an orthonormal basis
 - Again, try to determine the inverse of such matrices





Orthonormal vectors

- Recall that a collection of vectors is said to be *orthonormal* if
 - Any pair of different vectors in the collection are orthogonal
 - Their inner product is zero
 - Each of the vectors has a 2-norm equal to 1
- In $\mathbf{F}^n$, a collection of n orthonormal vectors forms a basis for that vector space
 - Recall that the inner product for complex vectors conjugates the entries of the first vector





Identity matrix

- Recall that an $n \times n$ identity matrix has 1s on the diagonal and zeros everywhere else

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$





Identity matrix

- You will note that the columns are the canonical basis for $\mathbf{R}^n$

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$





Permutation matrices

- Suppose we have an $n \times n$ matrix where the columns are a permutation (reordering) of the canonical basis vectors

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$





Permutation matrices

- Such matrices are called permutation matrices, but not only because the columns are a permutation of the canonical basis vectors

$$P_1 \mathbf{u} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_4 \\ u_1 \\ u_3 \end{pmatrix}$$

$$P_1 A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & \cdots & a_{4,n} \end{pmatrix} = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & \cdots & a_{4,n} \\ a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \end{pmatrix}$$





Permutation matrices

- Any matrix that permutes the entries of a vector or the rows of a matrix upon multiplication is called a permutation matrix
 - The columns consist of the canonical basis vectors
 - Each row has one “1” and each column has one “1”, while all other entries are zero
 - The identity matrix is a trivial permutation matrix
 - The row-operation matrix for swapping two rows $R_{R_i \leftrightarrow R_j}$ is a permutation matrix
 - The product of two permutation matrices is a permutation matrix

$$(P_2 P_1) \mathbf{u} = P_2 (P_1 \mathbf{u})$$





Permutation matrices

- The property of permutation matrices is that if $p_{i,j} = 1$ in a permutation matrix P , then
 - When determining the i^{th} entry of $P\mathbf{u}$, it selects the j^{th} entry of $\mathbf{u}$
 - When determining Row i of PA , it selects Row j of A
- For example, if $p_{1,3} = 1$,
then when the first entry of $P\mathbf{u}$ is calculated, we select u_3

$$P\mathbf{u} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{pmatrix}$$





Inverse of permutation matrices

- What is the inverse of a permutation matrix?
 - First, the inverse must be a permutation matrix
 - Given a permutation matrix P , if $p_{i,j} = 1$,
then when calculating the i^{th} entry of $P\mathbf{u}$, we select the j^{th} entry
- How do we reverse this operation?
 - When calculating the j^{th} entry of $P^{-1}\mathbf{u}$, we select the i^{th} entry
 - That is, if the entries of the inverse are $q_{i,j}$ then $q_{j,i} = 1$





Permutation matrices

- Thus, the inverses of each of the matrices we previously looked at include:

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad I_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad P_1^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad P_3^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$





The transpose

- The process of reflecting each entry through the diagonal is called taking the transpose of a given matrix A
 - The transpose is denoted as A^T

$$A = \begin{pmatrix} 12 & 2 & -1 & 5 \\ -4 & 14 & -3 & 4 \\ 3 & 8 & 23 & 0 \\ 0 & 7 & -6 & 1 \end{pmatrix} \quad A^T = \begin{pmatrix} 12 & -4 & 3 & 0 \\ 2 & 14 & 8 & 7 \\ -1 & -3 & 23 & -6 \\ 5 & 4 & 0 & 1 \end{pmatrix}$$





The transpose

- If P is a permutation matrix, then the inverse is the transpose

$$P^{-1} = P^T$$

- This is one case where it is not necessary to explicitly calculate the inverse,
but if you were, you would get the transpose





Orthonormal bases

- Recall that the identity matrix and permutation matrices are comprised of columns that make up a canonical basis for $\mathbf{R}^n$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- What if we had a different orthonormal basis?

$$\begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix} \quad \begin{pmatrix} 0.2 \\ 0.4 \\ 0.8 \\ -0.4 \end{pmatrix}, \begin{pmatrix} 0.56 \\ 0.52 \\ -0.56 \\ -0.32 \end{pmatrix}, \begin{pmatrix} -0.8 \\ 0.4 \\ -0.2 \\ -0.4 \end{pmatrix}, \begin{pmatrix} -0.08 \\ 0.64 \\ 0.08 \\ 0.76 \end{pmatrix}$$





Orthonormal bases

- Consider the matrices:

$$A = \begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.2 & 0.56 & -0.8 & -0.08 \\ 0.4 & 0.52 & 0.4 & 0.64 \\ 0.8 & -0.56 & -0.2 & 0.08 \\ -0.4 & -0.32 & -0.4 & 0.76 \end{pmatrix}$$

- What happens if we calculate $A^T A$ or $B^T B$?





Orthonormal bases

- Calculating $A^T A$:

$$\begin{pmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- We get the same result calculating AA^T :

$$\begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$





Orthonormal bases

- We get the same results calculating $B^T B$ and BB^T :

$$\begin{pmatrix} 0.2 & 0.4 & 0.8 & -0.4 \\ 0.56 & 0.52 & -0.56 & -0.32 \\ -0.8 & 0.4 & -0.2 & -0.4 \\ -0.08 & 0.64 & 0.08 & 0.76 \end{pmatrix} \begin{pmatrix} 0.2 & 0.56 & -0.8 & -0.08 \\ 0.4 & 0.52 & 0.4 & 0.64 \\ 0.8 & -0.56 & -0.2 & 0.08 \\ -0.4 & -0.32 & -0.4 & 0.76 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 0.56 & -0.8 & -0.08 \\ 0.4 & 0.52 & 0.4 & 0.64 \\ 0.8 & -0.56 & -0.2 & 0.08 \\ -0.4 & -0.32 & -0.4 & 0.76 \end{pmatrix} \begin{pmatrix} 0.2 & 0.4 & 0.8 & -0.4 \\ 0.56 & 0.52 & -0.56 & -0.32 \\ -0.8 & 0.4 & -0.2 & -0.4 \\ -0.08 & 0.64 & 0.08 & 0.76 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$





Orthogonal matrices

- A matrix that has columns that form an orthonormal basis is called an *orthogonal matrix*

Theorem

A real matrix is orthogonal if and only if the transpose is the inverse of the matrix





Orthonormal bases

- Consider this matrix in $\mathbf{C}^4$:

$$C = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$$

- Its columns form a basis for $\mathbf{C}^4$
 - Does $C^T C = I_4$?
- Hint: Remember the definition of the inner product for complex vector spaces





Relevance

- Calculating the inverse of a matrix can be expensive and is potentially numerically unstable
- If a matrix is orthogonal, then finding the inverse is easy and finding it is also numerically stable





Summary

- Following this topic, you now
 - Understand the definition of a permutation matrix
 - Know that the inverse of a permutation matrix is its transpose
 - Understand that if the columns of a matrix form an orthonormal basis, then we call the matrix orthogonal and the inverse is also its transpose
 - Every permutation matrix is an orthogonal matrix
 - Are aware that this makes calculating the inverse much easier
 - Know that we will discuss such matrices, including complex matrices, later on in the course





References

- [1] https://en.wikipedia.org/wiki/Invertible_matrix
- [2] https://en.wikipedia.org/wiki/Permutation_matrix
- [3] https://en.wikipedia.org/wiki/Orthogonal_matrix





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Keogan Larade for pointing out the incorrect transpose on Slide 12.





Colophon

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